A DYNAMIC ANALYSIS OF MOVING AVERAGE RULES

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ABSTRACT. The use of various moving average rules remains popular with financial market practitioners. These rules have recently become the focus of empirical studies. However there have been very few studies on the analysis of financial market dynamics resulting from the fact that some agents engage in such strategies. In this paper we seek to fill this gap in the literature by proposing a dynamic financial market model in which demand for traded assets has both a fundamentalist and a chartist component. The chartist demand is governed by the difference between a long run and a short run moving average. Both types of traders are boundedly rational in the sense that, based on a certain fitness measure, traders switch from a strategy with low fitness to the one with high fitness. We characterise first the stability and bifurcation properties of the underlying deterministic model via the reaction coefficient of the fundamentalists, the extrapolation rate of the chartists and the lag lengths used for the moving averages. By increasing the switching intensity, we then examine various rational routes to randomness for different, but fixed, long run moving averages. The price dynamics of the moving average rule is also examined and it is found that an increase of the window length of the long moving average can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour. The analysis of the corresponding stochastic model is able to explain various market price phenomena, including market crashes, price switching between different levels and price resistance.

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1. Introduction

Technical analysts, also known as "chartists", attempt to forecast prices by the study of patterns of past prices and a few other related summary statistics about security trading. Basically, they believe that shifts in supply and demand can be detected in charts of market movements. In an environment of efficient markets, technical trading rules should not be useful for generating excess returns. However, despite all the evidence presented in academic journals that security prices follow random walks, and consequently that these security markets are at least weak-form efficient, as defined by Fama (1970), the use of technical trading rules still seems to be widespread amongst financial market traders.

There have been various studies of the use and profitability of technical analysis. Taylor and Allen (1992) document the enduring popularity of the trading rules in their survey of currency traders in London. Of the respondents, 90% replied that technical trading rules are an important component of short-term investment strategies. Allen and Taylor (1990, 1992) suggest that this is an important finding given the apparent ability of exchange rates to move far from fundamentals over protracted periods of time, as documented by Frankel and Froot (1986, 1990). Earlier empirical literature on stock returns finds evidence that daily, weekly and monthly returns are predictable from past returns. Pesaran and Timmermann (1994, 1995) present further evidence on the predictability of excess returns on common stocks for the S&P 500 and Dow Jones Industrial portfolios, and examine the robustness of the evidence on the predictability of U.S. stock returns. Brock *et al* (1992) investigate the sources of the predictability by applying the bootstrap technique to two of the simplest and most popular trading rules, the moving average and the trading range break rules. They find that returns obtained from buy (sell) signals are not likely to be generated by four popular null models, these being the random walk, the AR(1), the GARCH-M and the EGARCH models. They document that buy signals generate higher returns than sell signals and the returns following buy signals are less volatile than returns following sell signals. This asymmetric nature of the returns and the volatility of the Dow series over the periods of buy and sell signals suggest the existence of nonlinearities in the data generation mechanism. Recent studies, such as Lo *et al* (2000), Boswijk *et al* (2000) and Goldbaum (2003), have also examined explicitly the profitability of technical trading rules and the implications for market efficiency. The profit generating potential of trading rules has also been scrutinised within the genetic programming framework by Neely *et al* (1997) and by the use of artificial neural networks by Gencay (1998) and Fernandez-Rodriguez et al (2000). Griffioen (2003) contains extensive statistical testing of the profitability of technical trading rules, after correcting for transaction costs and data snooping, of many stock market indices including the Dow Jones index.

Most of the cited research has focused on empirical studies. Furthermore, in terms of agents' actual demands (that are based on the various signals) and tests involving real world data, the hypothesis of profitability of trading rules is highly and ideally simplified. To apply the results in practice, the question as to how to determine the amount to buy/sell and how the market prices are affected following these buy/sell actions are not clear. There have been very few studies on the analysis of the type of financial market dynamics resulting from the fact that some agents engage in technical trading strategies. This paper seeks to fill this gap in the literature by proposing a market of financial market dynamics in which demand for traded assets has both a fundamentalist and a technical analyst component. The technical analysts are assumed to react to buy-sell signals generated by difference between a long run and a short run moving average. Both types of traders are boundedly rational in the sense that, based on a certain fitness measure, traders switch from strategies with low fitness to ones with high fitness.

The paper develops and analyzes a model in which individual boundedly rational behaviour leads to inefficiencies in an asset market which can be exploited through the use of various moving average rules. The main objectives of this paper are to analyze the stability properties of the model, particularly in relation to the moving average trading strategies, and the potential for the model to generate complex dynamics, and to examine the impact of the moving average trading rules on the market dynamics.

The plan of the paper is as follows. In the following section, we focus on one of the simplest cases when the fundamentalist demand is determined by mean reversion to the fundamental price, while the technical analyst demand is based on the sign of the difference of short and long moving averages, as in Chiarella (1992) and Brock and Hommes (1997, 1998). Based on certain fitness measures, such as observed differences in payoffs, the traders can make an endogeneous selection of which trading strategies to use, as in Blume et al (1994), Brock and Hommes (1997), Brock and LeBaron (1996) and Brown and Jennings (1989). Consequently, an adaptive heterogeneous asset pricing model with a market maker scenario is developed. In Section 3, the existence, local stability and bifurcation of the fundamental steady state, in terms of the reaction coefficient of the fundamentalists, the extrapolation rate of the technical analysts, the lag lengths used for the moving averages, and switching intensity, are analyzed when the lag lengths of the long moving average are small. The analysis, combined with some results on general window length for some special cases, gives us some important insights into the effect of increasing the length of the long moving average. In Section 4, by increasing the switching intensity among the two strategies, we examine numerically various rational routes to randomness, that is, bifurcation routes to complicated dynamical behavior as the intensity to switch strategies increases, for different, but fixed, long-run moving averages. The price dynamics induced by the moving average rule are then examined numerically in Section 5 and it is found that an increase of the window length of the long-run moving average can destabilize an otherwise stable system, leading to more complicated, even chaotic behaviour. Section 6 introduces a stochastic fundamental price and noise-trader demand processes, and examines the effect of these noise processes when the prices of the corresponding deterministic system are switching between bull and bear markets. This non-linear stochastic model illustrates a range of phenomena observed in real markets such as market crashes, price switching between different levels and price resistance. Section 7 concludes the paper.

2. An Asset Pricing Model under a Market Maker

Following the framework of Brock and Hommes (1998), this section sets up an asset pricing model with different types of heterogeneous traders who trade according to different trading rules, such as fundamental analysis and technical analysis. The market clearing price is arrived at via a market maker scenario in line with Day and Huang (1990) and Chiarella and He (2003*b*) rather than the Walrasian scenario used in Brock and Hommes (1998) and Chiarella and He (2002). Whilst the market maker and Walrasian auctioneer mechanisms are highly stylized accounts of how the market clearing price is arrived at, we feel that the former is closer to what is going on in real markets. To focus on the price dynamics of the trading rules, we motivate the excess demand functions of different types of traders by their trading rules directly, rather that the demand functions driven from utility maximization of their portfolio investment with both risky and risk-free assets (as for example in Brock and Hommes (1998) and Chiarella and He (2002, 2003*b*)).

Consider an asset pricing model with only one risky asset. Let P_t be the price (cum dividend) per share of the risky asset at time t. Let N be the total number of traders (assumed to be a constant) in the market, among which there are $N_{h,t}$ of type h traders at time t with $h = 1, 2, \cdots, H$ and $\sum_{h=1}^{H} N_{h,t} = N$. Then the market fractions of different types of traders at time t can be defined as

$$n_{h,t} = N_{h,t}/N, \qquad h = 1, 2, \cdots, H. \tag{2.1}$$

Let the excess demand for the risky asset of trader i at time t be $D_{i,t}$. Then the aggregate excess demand at time t is given by

$$D_t = \sum_{i=1}^{N} D_{i,t} = N \sum_{h=1}^{H} n_{h,t} D_t^h, \tag{2.2}$$

where D_t^h corresponds to the average excess demand function of type h traders. We assume that prices are set period by period via a market maker mechanism and adjusted according to the average excess demand D_t/N. Thus

$$P_{t+1} = P_t[1 + \sigma_\epsilon \epsilon_t] + \frac{\mu}{N} D_t x = P_t[1 + \sigma_\epsilon \tilde{\epsilon}_t] + \mu \sum_{h=1}^{H} n_{h,t} D_t^h, \tag{2.3}$$

where ϵ_t is an *i.i.d.* normally distributed random variable that captures a random excess demand process either driven by unexpected news about fundamentals, or representing noise created by *noise traders* with $\epsilon_t \sim \mathcal{N}(0, 1)$, $\sigma_\epsilon \geq 0$ is a constant and the parameter $\mu > 0$ measures the speed of price adjustment (or the aggregate risk tolerance) of the market maker to the excess demand.

For simplicity, we consider throughout this paper that there are only two types of traders: fundamentalists and technical analysts, who in fact are the most widespread types of traders in financial markets and whose trading strategies and excess demand functions are specified in the following discussion. We assume that there are $N_{f,t}$ fundamentalists and $N_{c,t}$ technical analysts at time t. Then the market fraction of fundamentalists and technical analysts at time t are given by, respectively

$$n_{f,t} = N_{f,t}/N, \qquad n_{c,t} = N_{c,t}/N. \tag{2.4}$$

The aggregate excess demand D_t at time t in (2.2) is then given by

$$D_t = N[n_{f,t}D_t^f + n_{c,t}D_t^c], \tag{2.5}$$

where D_t^f and D_t^c are the average excess demands of the fundamentalist and technical analyst to be defined, respectively. Set

$$m_t = n_{f,t} - n_{c,t},$$

so that $n_{f,t} = (1 + m_t)/2$ and $n_{c,t} = (1 - m_t)/2$. Following from (2.3)-(2.5), the market price of the risky asset is then determined by

$$P_{t+1} = P_t[1 + \sigma_\epsilon \epsilon_t] + \frac{\mu}{2}[(1 + m_t)D_t^f + (1 - m_t)D_t^c]. \tag{2.6}$$

Fundamentalists—The fundamentalists believe that the market price should be given by the fundamental price that they have estimated based on various types of fundamental information, such as earnings, exports, general economic forecasts and so forth. They buy/sell the stock when the current price is below/above the fundamental price. For simplicity, we first assume that[1] the fundamental price is a positive constant P^* and the average excess demand of the fundamentalists is given by[2]

$$D_t^f = \alpha(P^* - P_t), \tag{2.7}$$

where the parameter $\alpha > 0$ is a combined measure of the aggregate risk tolerance of the fundamentalists and their reaction to the *mis-pricing*.

Technical Analysts—Unlike the fundamentalists, the technical analysts trade based on charting signals generated from the costless information contained in the history of the price, such as moving averages and various other technical trading rules used in financial markets. The technical analyst average excess demand is here assumed to be based on signals generated by moving averages. [3] More precisely, a moving average

[1]A stationary random walk fundamental price will be introduced in Section 6

[2] Given an annual risk free rate r, the excess demand function in (2.7) should be formed by $D_t^f = \alpha[P^* - (1 + r/K)P_t]$, where K corresponds to the trading frequency per year. To characterize asset price dynamics at a high-frequency (such as $K = 250$ for daily) the risk-free rate per trading period r/K is very small, so here we simply treat it as zero.

[3]There is a great of practitioner literature the way moving average rules are used to generate buy/sell signals. See for instance Pring (1991) and Neely (1997).

of length k at time t is defined as

$$ma_t^k = \frac{1}{k}\sum_{i=0}^{k-1} P_{t-i}, \quad (k \geq 1).$$

A trading signal is defined as difference between a short-run moving average ma_t^S and a long-run moving average ma_t^L, namely

$$\psi_t^{S,L} = ma_t^S - ma_t^L, \tag{2.8}$$

where $L \geq S$ are positive integers. For the technical analysts, their average excess demands are assumed to be governed by

$$D_t^c = h(\psi_t^{S,L}), \tag{2.9}$$

where the function h has the general properties

$$h(0) = 0, \qquad h'(x) > 0, \qquad xh''(x) < 0. \tag{2.10}$$

This corresponds to the very popular technical trading rule based on the crossing of the long run and short run moving averages. By setting $S = 1$ we obtain the moving average rule whereby technical analysts wish to be long (short) when the current price is above (below) the moving average. For $S > 1$ we obtain the double moving average rule according to which the technical analysts go long (short) when the short run moving average moves above (below) the long run moving average. In this paper, we select

$$h(x) = \tanh(ax), \qquad a = h'(0) > 0.$$

We note that this form of technical analyst excess demand function allows us to capture some elements of the filtered moving average rules. This is so since, when a is small, the technical analysts initially react cautiously to the long/short signals, in a sense waiting to confirm the maintenance of the change in sign of the signal. In this way they minimize the costs incurred if the signal changes frequently in a short time period. Also, the fact that $-1 < h(x) < 1$ can be used to capture the limited long/short positions, risk averting and traders' budget constraints.

Fitness Measure and Population Evolution—In order to introduce the adaptiveness of agents, we follow the mechanism of Brock and Hommes (1998) and define the fitness functions $\pi_{f,t}, \pi_{c,t}$ as their realized capital gains[4]:

$$\pi_{f,t} = D_{t-1}^f (P_t - P_{t-1}) - C_f, \qquad \pi_{c,t} = D_{t-1}^c (P_t - P_{t-1}) - C_c, \tag{2.11}$$

where $C_f, C_c \geq 0$ are the costs of their strategies. Then the population fractions are assumed to be updated by the well known *discrete choice model* or 'Gibbs' probabilities (e.g. Manski and McFadden (1981))

$$n_{f,t} = \frac{e^{\beta U_{f,t}}}{e^{\beta U_{f,t}} + e^{\beta U_{c,t}}}, \qquad n_{c,t} = \frac{e^{\beta U_{c,t}}}{e^{\beta U_{f,t}} + e^{\beta U_{c,t}}}, \tag{2.12}$$

[4]As indicated in footnote 2, we assume the risk free rate for the trading period is zero.

where

$$U_{f,t} = \pi_{f,t} + \eta U_{f,t-1}, \qquad U_{c,t} = \pi_{c,t} + \eta U_{c,t-1}, \tag{2.13}$$

and $\eta \in [0,1]$ measures the memory of the cumulated fitness function and $\beta \geq 0$ measures the switching intensity among the two strategies. In particular, if $\beta = 0$, there is no switching between strategies among agents. See Brock and Hommes (1998) for a more extensive discussion of this switching mechanism.

A Complete Asset Pricing Model—It follows from (2.5)-(2.6) that the market price of the risky asset is determined according to

$$P_{t+1} = P_t[1 + \sigma_\epsilon \tilde{\epsilon}_t] + \frac{\mu}{2}\big[(1+m_t)\alpha(P^* - P_t) + (1-m_t)h(\psi_t^{S,L})\big] \tag{2.14}$$

and, from (2.11)-(2.12), that the difference of population fractions m_t evolves according to

$$m_t = \tanh\big[\frac{\beta}{2}(U_t - C)\big], \tag{2.15}$$

where $C = C_f - C_c \geq 0$, $\mu \geq 0$ measures the speed of price adjustment of the market maker based on the excess demand, and

$$U_t = [D_{t-1}^f - D_{t-1}^c][P_t - P_{t-1}] + \eta U_{t-1}, \tag{2.16}$$

By setting $\sigma_\epsilon = 0$, the nonlinear stochastic dynamical system (2.14)-(2.16) becomes a nonlinear deterministic system where the price follows

$$P_{t+1} = P_t + \frac{\mu}{2}\Bigg[(1+m_t)\alpha(P^* - P_t) + (1-m_t)h(\psi_t^{S,L})\Bigg]. \tag{2.17}$$

In general system (2.15)-(2.17) is an $L+2$ dimensional non-linear difference system. We seek principally to understand how its dynamic behaviour is affected by the reaction coefficient of the fundamentalists, the excess demand function of the technical analysts, the lengths used for the moving averages, and the switching intensity.

3. Stability and Bifurcation Analysis

In this section, we consider the price dynamics of the deterministic system (2.15)-(2.17). We first state the following result on the existence of the unique steady state and the corresponding characteristic equation.

Proposition 3.1. *For the deterministic system (2.15)-(2.17), assume $\eta \in [0,1)$. Then there exists a unique steady state $(P_t, m_t, U_t) = (P^*, m^*, 0)$, where P^* is the constant fundamental price and $m^* = \tanh(-\beta C/2)$. In addition, the characteristic equation of this steady state is given by*

$$\Gamma(\lambda) = \lambda(\lambda - \eta)\Gamma_{S,L}(\lambda) = 0, \tag{3.1}$$

where

$$\Gamma_{S,L}(\lambda) \equiv \lambda^L - (1-\bar{\alpha})\lambda^{L-1} - \bar{a}(\frac{1}{S} - \frac{1}{L})(\lambda^{L-1} + ... + \lambda^{L-S}) + \frac{\bar{a}}{L}(\lambda^{L-S-1} + \cdots + \lambda + 1) \tag{3.2}$$

and

$$\bar{\alpha} = \alpha\mu(1+m^*)/2, \quad \bar{a} = a\mu(1-m^*)/2. \tag{3.3}$$

Proof. See Appendix A.1 □

The parameter $\bar{\alpha}$ measures the combined effect of the speed of price adjustment of the market maker toward the aggregate excess demand (μ), the speed of current price adjustment of the fundamentalists towards their expected fundamental price (α), and the market equilibrium fraction (m^*). The parameter $\bar{a}$ measures the combined effect of the speed of price adjustment of the market maker (μ), the extrapolation rate of the technical analysts to the difference of short and long run moving averages (a), and the equilibrium market fraction (m^*). Obviously, $m^* = 0$ when $C = 0$. The parameters $\bar{\alpha}$ and $\bar{a}$ turn out to play important roles in the characterisation of the dynamic behaviour of the model.

We now analyse the local stability of the unique steady state and its bifurcation properties. Given the structure of equation (3.1), the local stability and bifurcations are determined by the eigenvalues of $\Gamma_{S,L}(\lambda) = 0$. For simplicity, in the following discussion, we concentrate on the case $S = 1$ and $L \geq 1$. For general $L \geq 1$, we first obtain the following result.

Lemma 3.2. *Let $S = 1$ and $L > 1$ in the characteristic equation (3.1).*

(i). *If $\bar{\alpha} = 1 + \bar{a}$, then the eigenvalues λ_i of $\Gamma_{1,L}$ satisfy $|\lambda_i| < 1$ if and only if $0 < \bar{a} < L$. In addition, for $\bar{a} = L$, the λ_i satisfy $\lambda_i \neq 1$ and $(1-\lambda_i^L)/(1-\lambda_i) = 0$.*

(ii) *A necessary condition for $|\lambda_i| < 1$ for all i is*

$$0 < \bar{a} < L, \qquad 0 < \bar{\alpha} < \begin{cases} 2 + \bar{a} & \textit{for } L = 2l; \\ 2 + \frac{L-1}{L}\bar{a} & \textit{for } L = 2l+1. \end{cases} \tag{3.4}$$

Proof. See Appendix A.2. □

The above Lemma 3.2 leads to the following corollary.

Corollary 3.3. *Consider the characteristic equation (3.1) with $S = 1$ and $L \geq 1$,*

- *if $\bar{\alpha} = 1 + \bar{a}$, then the steady state price P^* is locally stable for $0 < \bar{a} < L$. In addition, at $\bar{a} = L$, there occurs a $1 : L+1$ resonance bifurcation*[5].

[5]Resonance bifurcations occur when the eigenvalues lie on the unit circle. When $\bar{a} = L$, the eigenvalues are given by $\lambda_k = e^{2k\mu\pi i}$ with $k = 1, 2, \cdots, L$ and $\mu = 1/(L+1)$. Geometrically, the L eigenvalues correspond to the $L+1$ unit roots distributed evenly on the unit circle, excluding $\lambda = 1$. When $L = 1$, a flip or period-doubling bifurcation occurs. When $L = 2$, according to Kuznetsov (1995), the bifurcation is known as a 1:3 strong resonance, which may lead to two sets of period three cycles with one set stable and other set unstable (see Chiarella and He (2000) for more details). For $L \geq 2$, according to Sonis

- *a necessary condition for the steady state price to be stable is given by (3.4). In addition, a flip bifurcation occurs at* $\bar{\alpha} = 2 + \bar{a}$ *for even* L *and* $\bar{\alpha} = 2 + \bar{a}(L-1)/L$ *for odd* L.

The first result in Corollary 3.3 indicates the stability of the fundamental price along the line $\bar{\alpha} = 1 + \bar{a}$ only, as indicated in Fig. 3.1. This special case however has two implications. First, along the line, the stability region is proportionally enlarged as lags for the long moving average L increase. Secondly, for fixed lag L, the stability line segment $\bar{\alpha} = 1 + \bar{a}$ for $0 < \bar{a} < L$ is part of the stability region for the general lag length L on the parameter $(\bar{\alpha}, \bar{a})$ parameter plane. Consequently, we may conjuncture that the stability region is enlarged as lag L increases. However this conjuncture is in general not true and this becomes clear from the following theoretical results for cases of $L = 1, 2, 3$ and 4 within this section and numerical results for higher lags L in Sections 4 and 5. The second result in Corollary 3.3 gives us necessary stability boundaries for $\bar{a}$ and $\bar{\alpha}$ and they are indicated by the two dotted lines in Fig. 3.1. For general lag L, we have the following result that gives more precisely common stability region D_S for any lags.

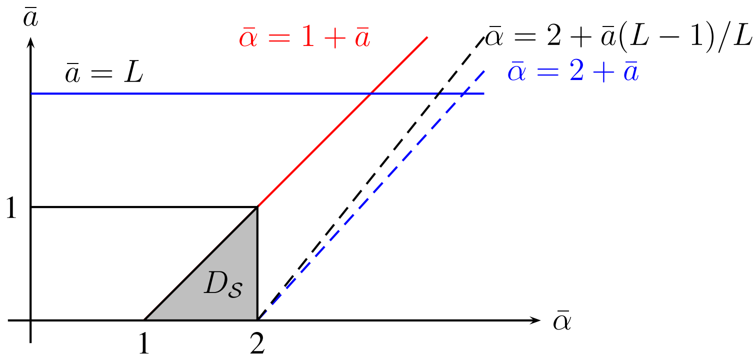

FIGURE 3.1. Common stability region D_S and necessary stability boundaries $\bar{a} = L$ and $\bar{\alpha} = 2+\bar{a}$ for even lag L and $\bar{\alpha} = 2+\bar{a}(L-1)/L$ for odd lag L for $S = 1$ and general lag $L > 1$.

Proposition 3.4. *Consider the characteristic equation (3.1) with* $S = 1$ *and* $L > 1$. *If* $1 + \bar{a} < \bar{\alpha} < 2$ *then* P^* *is locally asymptotically stable (LAS).*

Proof. See Appendix A.3 □

(2000), the bifurcation is accompanied by $1 : L + 1$ periodic resonances. For $L_1 = L_2 = L = 3, 4$, instability of the steady state leads to 1:4 and 1:5 periodic resonance bifurcations, respectively, and similar dynamics to the 1:3 resonance bifurcation are also found. Theoretical analysis of such types of bifurcation of higher dimensional discrete systems can be exceedingly complicated and is not yet completely understood, (see Example 15.34 in Hale and Kocak (pp. 481-482, (1991))) The stability region and necessary stability boundaries in terms of parameters $(\bar{\alpha}, \bar{a})$ given by Proposition 3.4 are plotted in Fig. 3.1. The region $D_{\mathcal{S}}$ corresponds to the stability region defined by Proposition 3.4. Note that the stability condition holds for all L, indicating that the region $D_{\mathcal{S}}$ is in fact the common stability region for all L. This is further verified by the following results where stability and bifurcation are analysed for $L = 1, 2, 3$ and 4.

For $L = 1, 2, 3$ and 4, the following proposition describes explicitly the regions of LAS in the $(\bar{\alpha}, \bar{a})$ plane and the bifurcation behaviour at the boundaries of those regions where local asymptotic stability turns to instability.

Proposition 3.5. *Consider the characteristic equation (3.1). For $S = 1$ and $L = 1, 2, 3, 4$, the local stability and bifurcation of the fixed point P^* can be described as follows:*

(i) *For $L = 1$, P^* is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{11}(\bar{\alpha}, \bar{a}) \equiv \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < 2, 0 < \bar{a}\}.$$

In addition
- *a flip bifurcation occurs when $\bar{\alpha} = 2$, and*
- *a saddle-node bifurcation occurs when $\bar{\alpha} = 0$.*

(ii) *For $L = 2$, P^* is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{12}(\bar{\alpha}, \bar{a}) \equiv \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \bar{a} + 2, 0 < \bar{a} < 2\}.$$

Furthermore,
- *a saddle-node bifurcation occurs when $\bar{\alpha} = 0$,*
- *a Hopf bifurcation occurs when $\bar{a} = 2$, and*
- *a flip bifurcation occurs when $\bar{\alpha} = \bar{a} + 2$.*

(iii) *For $L = 3$, P^* is LAS if*

$$(\bar{\alpha}, \bar{a}) \in D_{13}(\bar{\alpha}, \bar{a}) \equiv \{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \frac{2}{3}\bar{a} + 2, \bar{a}(2 - \bar{\alpha} + \bar{a}) < 3\}.$$

Furthermore,
- *a saddle-node bifurcation occurs when $\bar{\alpha} = 0$,*
- *a Hopf bifurcation occurs when $\bar{a}(2 - \bar{\alpha} + \bar{a}) = 3$, and*
- *a flip bifurcation occurs when $\bar{\alpha} = \frac{2}{3}\bar{a} + 2$.*

(iv) *For $L = 4$, P^* is LAS if*

$$\begin{aligned}(\bar{\alpha}, \bar{a}) \in D_{14}(\bar{\alpha}, \bar{a}) \equiv &\{(\bar{\alpha}, \bar{a}); 0 < \bar{\alpha} < \frac{3}{4}\bar{a} + 2, 0 < \bar{a} < 4, \\ &(5\bar{a} - 4\bar{\alpha})(4 + \bar{a})^2 < \bar{a}(8 + 3\bar{a} - 4\bar{\alpha})^2\}.\end{aligned}$$

In addition, a flip bifurcation occurs when $\bar{\alpha} = \frac{3}{4}\bar{a} + 2$.

Proof. See Appendix A.4. □

Consider the case $S = L = 1$ for which the stability region is D_{11}. Obviously, the stability condition is independent of $\bar{a}$, as shown in Fig. 3.2(a). In this case, the

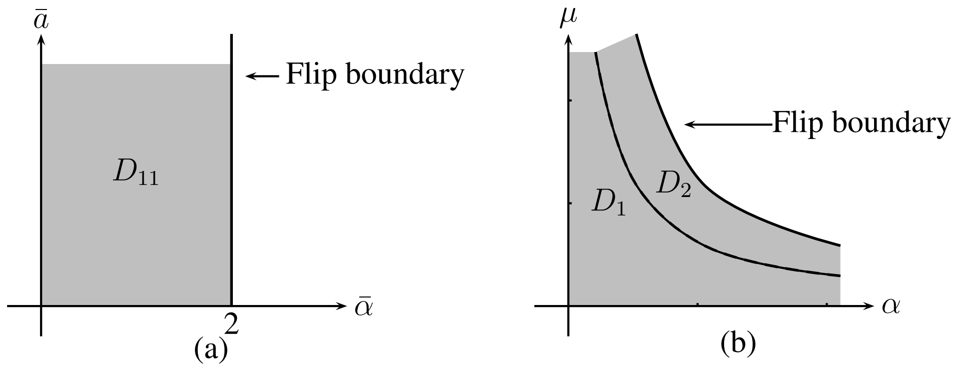

FIGURE 3.2. Stability region D_{11} and bifurcation boundary in the $(\bar{\alpha}, \bar{a})$ plane (a), and the (α, μ) plane (b), where $D_{11} = D_1 \cup D_2$ and $\lambda \in (0, 1)$ in D_1, $\lambda \in (-1, 0)$ in D_2.

technical analysts play no role on the market price and hence $m_t = 1$. Consequently, the price equation is simplified to $P_{t+1} - P^* = [1 - \alpha\mu](P_t - P^*)$. Hence the stability condition $0 < \bar{\alpha} = \alpha\mu < 2$ can be expressed in terms of the speed of the price adjustment of the fundamentalists towards the fundamental price (α) and the speed of price adjustment of the market maker (μ). The stability region in terms of the parameters α and μ is plotted in Fig. 3.2(b), indicating that the stability of the steady state price P^* is maintained only when the reaction speeds from both the fundamentalists and the market maker are balanced in a certain way. Note that, when $\alpha\mu = 1$, the prices stay at the constant steady state price P^*. The stability region D_{11} is then divided into two regions $D_1(\alpha\mu < 1)$ and $D_2(1 < \alpha\mu < 2)$. On the one hand, the eigenvalue $\lambda = 1 - \alpha\mu$ is positive for $(\alpha, \mu) \in D_1$ and negative for $(\alpha, \mu) \in D_2$. Consequently, relative to the steady state price, the returns of the market price P_t are positively (negatively) correlated in the region D_1 (D_2). On the other hand, in the region D_1 the market price is under-adjusted (or under-reacted) by either the market maker or the fundamentalists, while in the region D_2 the market price is over-adjusted (or over-reacted) by both the market maker and the fundamentalists. We thus call D_1 (D_2) a region of under-reaction (over-reaction) from the point of view of either the market maker or the fundamentalists. In addition, $\bar{\alpha} = 2$ leads to a flip bifurcation, resulting from overreaction of either the market maker or the fundamentalists.

Consider next the case $L = 2$. The stability region D_{12} and bifurcation boundaries are plotted in Fig. 3.3(a) in the $(\bar{\alpha}, \bar{a})$ parameter plane. The stability region D_{12} can be divided into three regions $D_{12} = D_1 \cup D_2 \cup D_3$ with both the eigenvalues $\lambda_{1,2}$ being positive in D_1, negative in D_2, and complex in D_3. Along the boundary between D_1, D_2 and D_3, we have double real eigenvalues. The Hopf bifurcation boundary is defined by $\bar{a} = 2$ and $\bar{\alpha} \in (0, 4)$. The nature of the Hopf bifurcation is determined by the value ω of the complex eigenvalues $\lambda_{1,2} = e^{\pm 2\pi i \omega}$, and hence the value of $\rho \equiv 2\cos(2\pi\omega)$ (see Chiarella and He (2003*a*) for detailed discussion on how the

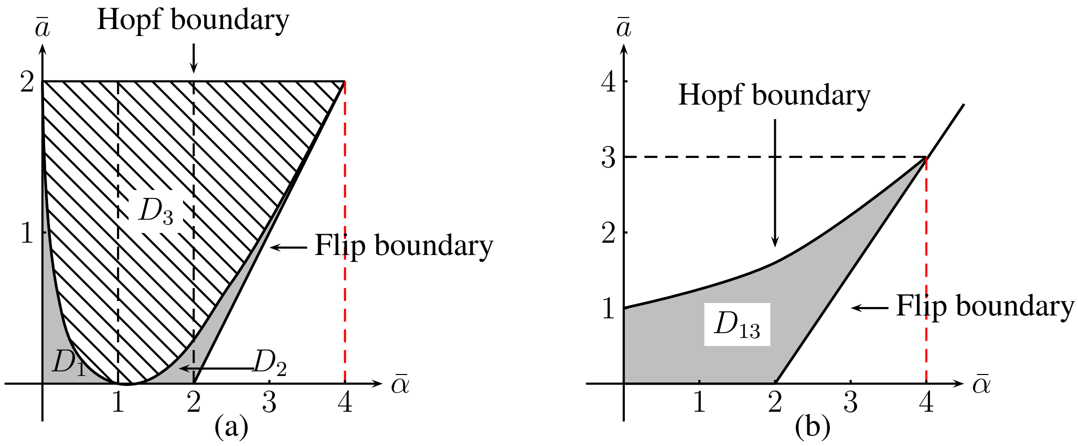

FIGURE 3.3. (a) Stability region $D_{12} = D_1 \cup D_2 \cup D_3$ and bifurcation boundaries for $S = 1$ and $L = 2$; (b) Stability region D_{13} and bifurcation boundaries for $S = 1$ and $L = 3$.

nature of the bifurcation is related to the values of ρ). It can be verified that, along the Hopf bifurcation boundary, $\rho = 2 - \bar{\alpha} \in [-2, 2]$ for $\bar{\alpha} \in [0, 4]$.

In the case $L = 3$. The stability region D_{13} and the bifurcation boundaries are plotted in Fig. 3.3(b) on the $(\bar{\alpha}, \bar{a})$ parameter plane. Different from the cases $S = 1$ and $S = 2$, the Hopf bifurcation now depends on both parameters $\bar{\alpha}$ and $\bar{a}$. The nature of the Hopf bifurcation is determined by $\rho \equiv 2\cos(2\pi\omega) = 3/\bar{a} - 1 \in [0, 2]$ for $\bar{a} \in 1, 3]$.

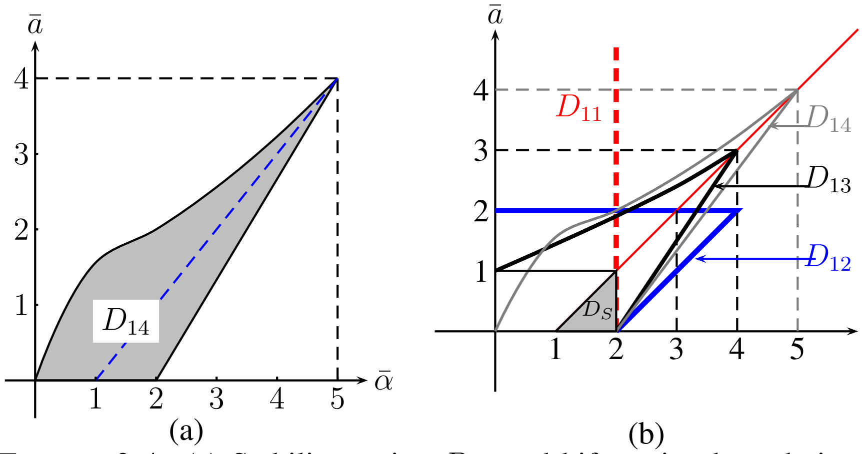

FIGURE 3.4. (a) Stability region D_{14} and bifurcation boundaries for $S = 1$ and $L = 4$; (b) Comparison of stability regions and bifurcation boundaries D_{1L} for $L = 1, 2, 3, 4$.

In the case $L = 4$, the stability region D_{14} can be plotted in Fig. 3.4(a) on the $(\bar{\alpha}, \bar{a})$ parameter plane. A comparison of the stability regions D_{1L} for $L = 1, 2, 3$ and 4 is plotted in Fig. 3.4(b), and leads to the following observations:

- As L increases, the stability region for the parameter $\bar{a}$ becomes smaller for smaller values of $\bar{\alpha}$ (say $\bar{\alpha} < 2$), and is enlarged for larger values of $\bar{\alpha}$.
- The steady state can only be locally stable when either the fundamentalists reduce their speed of price adjustment towards their expected fundamental price and the technical analysts extrapolate the difference of the moving averages weakly, or the reactions of the technical analyst and fundamentalist are balanced in a certain way (that is, the parameters $\bar{\alpha}$ and $\bar{a}$ are near the line $\bar{\alpha} = 1 + \bar{a}$, as indicated in Lemma 3.2 and Fig. 3.4(b)).
- Based on the analytical results for $L = 1, 2, 3, 4$ and Corollary 3.3 and Proposition 3.4 for general L, we conjecture that: *as lag L increases, the stability region tends to shrink towards, but stretch along, the line $\bar{\alpha} = 1 + \bar{a}$ with common stability region D_S.*

Given a large variety of moving average rules used in financial markets and the difficulty of eigenvalue analysis for high-order characteristic equations, it is not clear how different moving average rules influence the stability of the steady state price and types of bifurcation that may occur. However the analysis has given some important insights into the fact that local asymptotic stability depends on some subtle balance between the reactions of fundamentalists and technical analysts. We are also able to make a conjecture about the general effect of the lag length of the long moving average. This conjecture is partly verified by the numerical simulations in the following sections. In the following section, we examine numerically some rational routes to randomness when agents' switching intensity increases for different moving average rules.

4. RATIONAL ROUTES TO RANDOMNESS

Brock and Hommes (1997, 1998) have proposed simple *Adaptive Belief System* to model economic and financial markets, where agents base decisions upon predictions of future values of endogenous variables whose actual values are determined by equilibrium equations. Agents adapt their beliefs over time by choosing from different predictors or expectations functions, based upon their past performance as measured by realized profits. Brock and Hommes (1998) show that, as the intensity of the switching to better strategies increases, the model is able to generate the entire "zoo" of complex behaviour from local stability to high order cycles and even chaos and this is the so-called *Rational Routes to Randomness (RRR for short).* In spirit of RRR, in this section, we consider the effect of the switching intensity on the price dynamics of the deterministic system (2.15)-(2.17). In order to see the effect of different long-run moving averages, we choose $S = 1$ and consider two extreme cases of the long-run moving average $L = 4$ and $L = 100$, respectively. In both cases, we select a fixed set of parameters as follows:

$$\alpha = 1, \mu = 2, \eta = 0.2, a = 1, C = 1. \tag{4.1}$$

Note that for $\beta = 0$, it follows from (3.3) and (4.1) that $\bar{\alpha} = 1$ and $\bar{a} = 1$.

4.1. **The Case $L = 4$.** For $\beta = 0$, the fundamental price P^* is locally stable. As β increases, $\bar{a}$ increases and $\bar{\alpha}$ decreases. It then follows from Proposition 3.5(iv) that the fundamental price becomes unstable as the switching intensity increases. This is verified by numerical simulations. To illustrate the effect of the switching intensity β on the price and population dynamics, we include phase plots, in terms of (P_t, m_t), for different values of $\beta = 0.2, 0.3, 0.49, 0.52, 0.555$ and 0.57 in Fig. 4.1. It is found that, once the fundamental price P^* becomes unstable, the solutions converge to *figure-eight shaped* attractors for low switching intensity(e.g. the case $\beta = 0.2$ and 0.3). As the switching intensity increases, the figure-eight shaped attractor grows initially (for $\beta = 0.3, 0.4$) and then stretches to a *scissors-shaped* attractor (for $\beta = 0.49$). As the intensity increases further, the simple attractor becomes more complicated (for $\beta = 0.52$) and eventually leads to strange attractors (for $\beta = 0.555$ and 0.57). The bifurcation diagram of the price and the corresponding Lyapunov exponent with respect to the switching intensity parameter β are plotted in Fig. 4.3. One can see that the market price variation increases as the switching intensity increases. It is interesting to note that these patterns are similar to the rational routes to randomness studied extensively in Brock and Hommes (1997, 1998). A common interesting feature displayed in Fig. 4.1 is that all the attractors are symmetric about the constant fundamental price. This feature is also shared in most cases for general lag L. A much more extensive analysis would be required to determine the nature of the mechanism generating such a feature, it may be caused by either the Hopf bifurcations or the special structure of the model.

To illustrate the time series behind the interesting phase plots in Fig. 4.1, the price time series for $\beta = 0.2, 0.49, 0.52$ and 0.57 are plotted in Fig. 4.2 over the first 500 trading periods. It is found that, as the switching intensity increases, the prices oscillate first around the fundamental price periodically or quasi-periodically and then irregularly, pushing the prices up and down with period closely related to the lag length (this becomes clear from the following discussion of the case $L = 100$). Also, the price becomes more volatile (e.g. the case $\beta = 0.52$ and 0.57).

4.2. **The Case $L = 100$.** For $\beta = 0$, we conjectured earlier that the fundamental price P^* is unstable for large L with the selected parameters and this is confirmed by numerical simulations. To illustrate the effect of the switching intensity β on the price and population dynamics with a long moving average of $L = 100$, in contrast to the case $L = 4$ in the previous discussion, we include phase plots, in terms of (P_t, m_t), for different values of $\beta = 0.05, 0.1, 0.2, 0.3, 0.35, 0.42, 0.45, 0.46$ and 0.4652 in Fig. 4.4. As β increases, the attractor starts with narrow *figure-eight shapes* (for $\beta = 0.05$ and 0.1) and is then stretched (or extrapolated) by the technical analysts towards the extreme high/low price levels (for $\beta = 0.2$). The closed attractors are then broken down to *Lorenz-like attractors* of the 3-dimensional continuous Lorenz system (see Peitgen *et al* (1992)) for β between 0.3 and 0.35, which is not observed for the case $L = 4$. As the switching intensity increases further, the *Lorenz-like* attractors merge into one connected strange attractor (for $\beta = 0.42$) and then to strange attractors (for

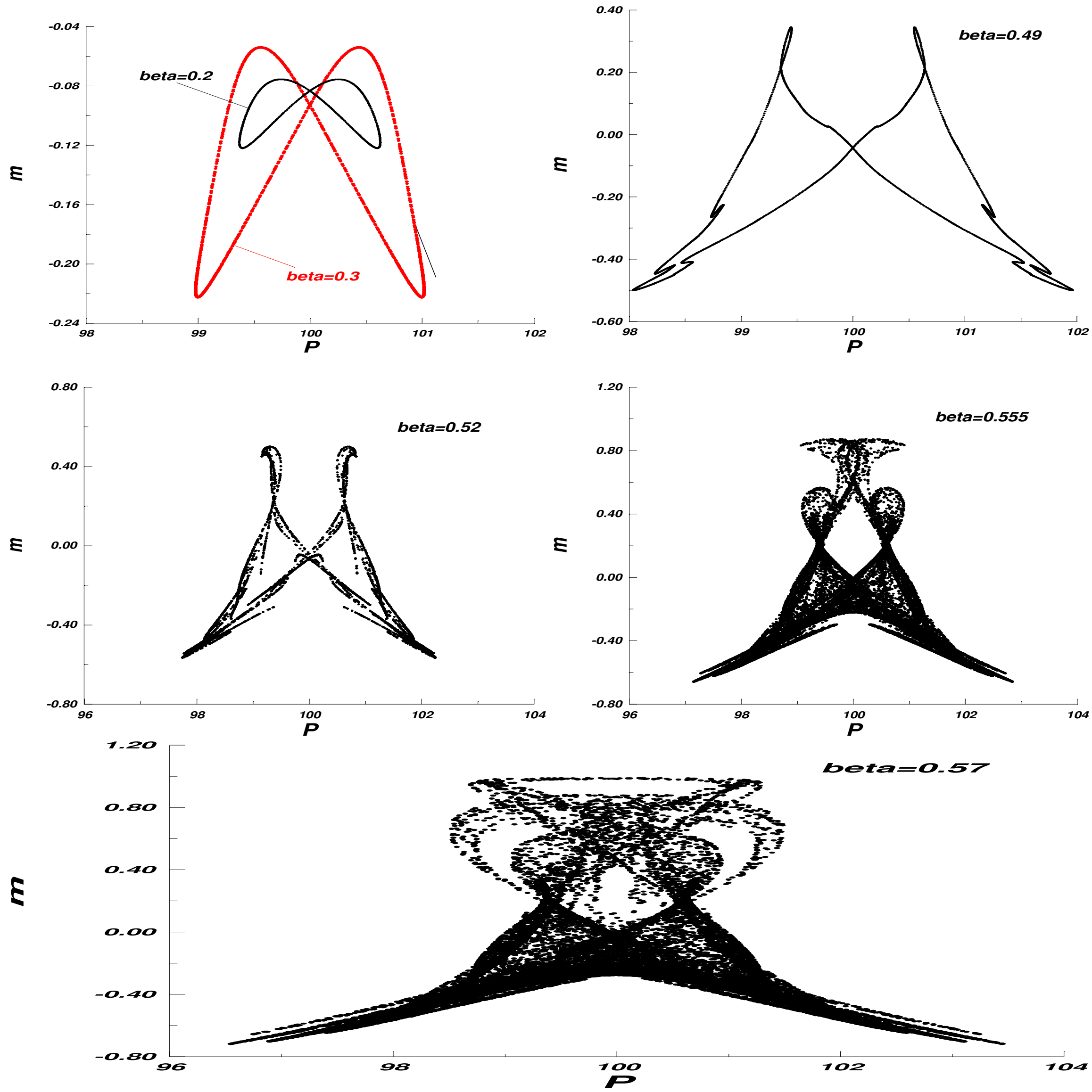

FIGURE 4.1. Phase plots of (m_t, P_t) for fixed $L = 4$ and various $\beta = 0.2, 0.3, 0.49, 0.52, 0.555$ and 0.57.

$\beta = 0.45, 0.46$ and 0.4652). The bifurcation diagram and the corresponding Lyapunov exponent with respect to the switching intensity parameter β are plotted in Fig. 4.6. Similar to the case of $L = 4$, as the switching intensity increases, the volatility of both price and population increases.

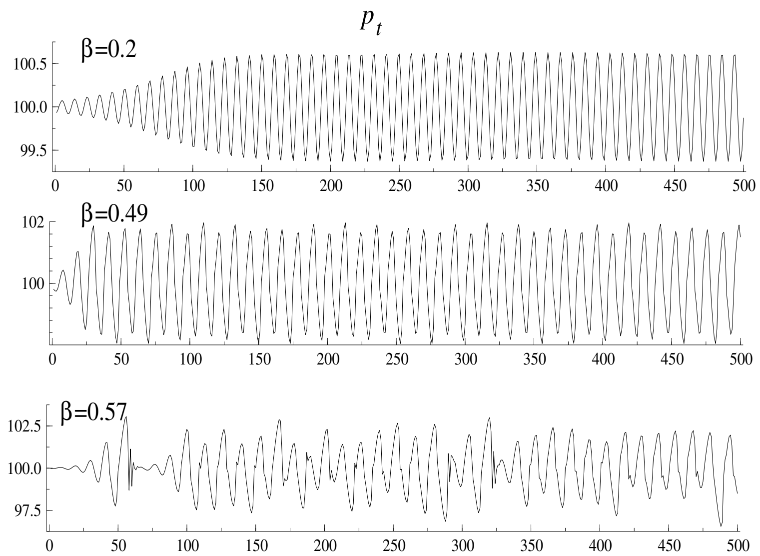

FIGURE 4.2. Price time series for $\beta = 0.2$ (a), 0.49 (b) and 0.57 (c) with fixed $L = 4$.

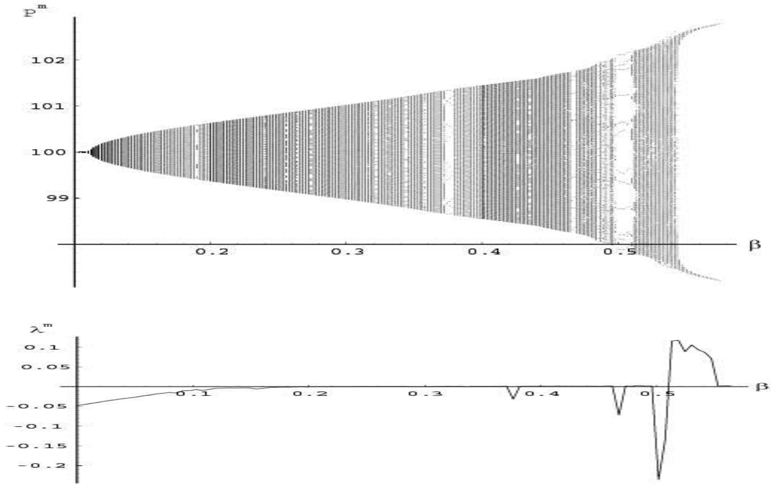

FIGURE 4.3. Bifurcation diagram and the Lyapunov exponent in terms of the parameter β with fixed $L = 4$.

The corresponding price time series are illustrated for $\beta = 0.1, 0.3, 0.35, 0.42$ and 0.46 in Fig. 4.5. One can see that an increase of the switching intensity can generate very interesting price patterns when $L = 100$, compared with the case of $L = 4$.

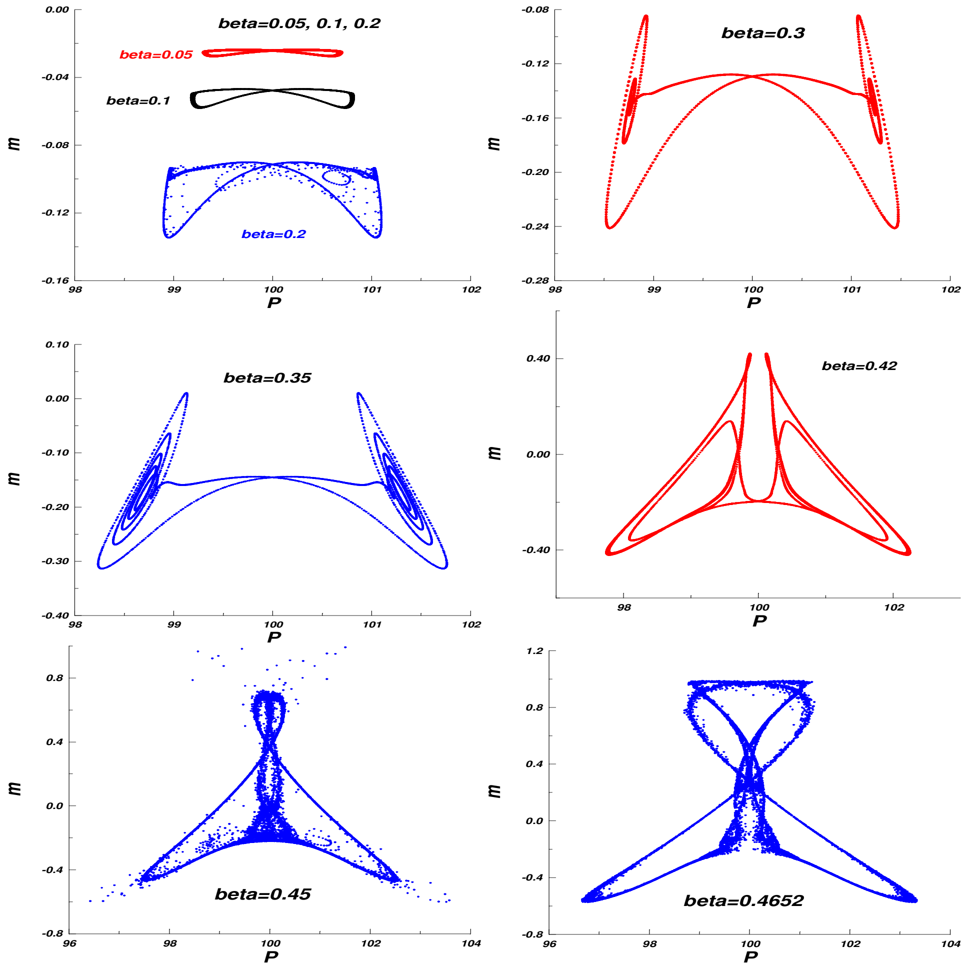

FIGURE 4.4. Phase plots of (m_t, P_t) for fixed $L = 100$ and various $\beta = 0.05, 0.1, 0.2, 0.3, 0.35, 0.42, 0.45$ and 0.4652.

With lower switch intensity ($\beta = 0.1$), the fundamental price is unstable and extrapolation of the price trend by the technical analysts pushes the price away from the fundamental price. Because of their limited long/short position[6], their fitness or utility becomes smaller when they reach their limit position. This leads traders to switch back to the fundamental strategy, bringing price back towards the fundamental price. Because of the increase of the fitness of the technical analysts, the price is pushed further beyond the fundamental price to the opposite extreme level. As the switching intensity increases (for $\beta = 0.3, 0.35$), such switching from high/low extreme to low/high extreme happens very quickly. At the same time, the price becomes more volatile.

[6]Recall that the form of the technical analysts demand function (2.10) implies limited long/short positions

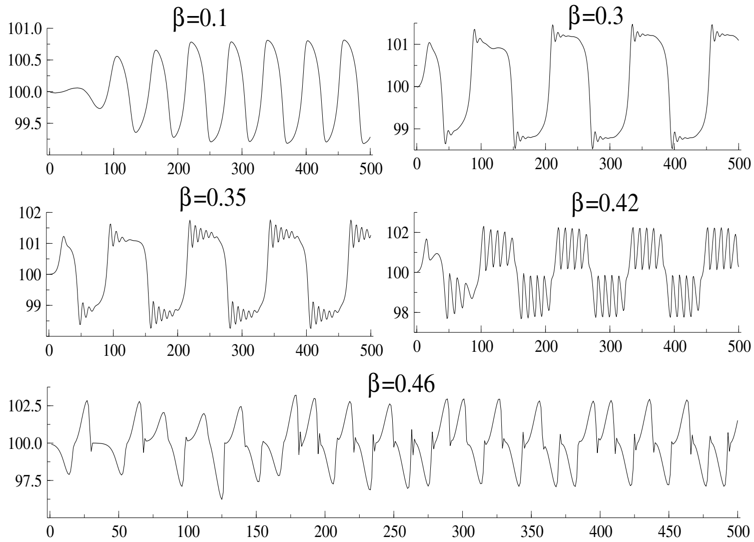

FIGURE 4.5. Price time series for $\beta = 0.1$ (a), 0.3 (b), 0.35 (c), 0.42 (d) and 0.46 (e).

This result can be used to explain regular boom and bear markets. As the intensity increase further, the regular switching pattern of the price between two extreme levels is destroyed, leading to highly volatile price patterns (for $\beta = 0.46$).

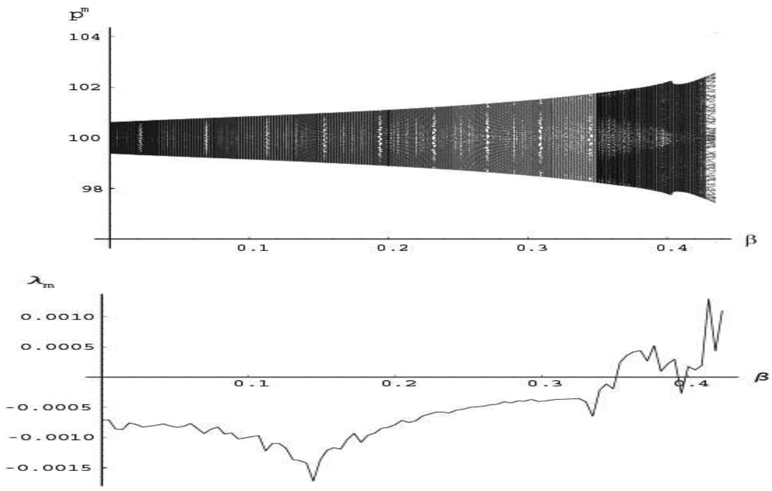

FIGURE 4.6. Bifurcation diagram and the Lyapunov exponent in terms of the parameter β with fixed $L = 4$.

It is interesting to note the different rational routes to randomness for $L = 4$ and $L = 100$. For $L = 4$, the strange attractors (in terms of the phase plots) become more dense as the switching intensity increases and the frequency of oscillation of the price series are very high (see Fig. 4.2). However, for $L = 100$, the time periods during which the price stays at either high or low levels are prolonged and prices become even volatile at the extreme levels (see Fig. 4.5). Correspondingly, the strange attractors concentrate more at the extreme levels and become less dense within the attractors (see Fig. 4.4). This phenomenon of the price switching between upper and lower levels gives some economic basis to the notion of upper and lower resistance levels that are frequently discussed in the practitioner literature on technical analysis (see e.g. Pring (1991)).

5. Dynamics of Long-Run Moving Average

In this section, we consider the effect of the long-run moving average on the price dynamics of the deterministic system (2.15)-(2.17). For comparison, we select a fixed set of parameters as follows:

$$\alpha = 1, \mu = 2, \beta = 0.4, \eta = 0.2, a = 1, C = 0. \tag{5.1}$$

It follows from (3.3) that $\bar{\alpha} = 1$ and $\bar{a} = 1$. Hence the fundamental price is locally stable for $L = 2, 3, 4$ and unstable for $L \geq 5$. Fig. 5.1 illustrates how the phase plot (in terms of (P_t, m_t)) changes as the lag L increases.

For $L = 5$, the attractor is given by a *figure-eight shaped* closed orbit with small price variation (about 1% of the fundamental price level) and there is a tendency among the traders to switch from the fundamentalist analysis to the technical analysts. For $L = 8$, the size of the attractor is enlarged, implying that the deviations of both price and population from the fundamental value, which is $P_t = 100, m_t = 0$, is enlarged. Hence an increase in the moving average window L destabilizes the price dynamics. This destabilizing effect becomes more significant when L is increased further to $L = 9, 10, 50$ and the price dynamics become even more complicated for $L = 90$ and 100, as indicated by the phase plots in Fig. 5.1.

In order to give more insights into such destablizing effects of the long-run moving average, let us examine the time series in Fig. 5.2. It is found that, following the cross over of the long moving average and the market price, both the technical analysts and fundamentalists take the same long/short position initially, but soon after they take opposite positions. This helps to accentuate either the up or the down trend, pushing the price to either a higher or a lower level initially, but soon after, their different positions slow down the trend built up initially and bring the price back towards the fundamental price level. The time period taken for the recent price back towards the fundamental price is proportional to the lag L. When the lag L for the moving average is small, the reversion back to the fundamental happens quickly. As L increases, this reversion takes a longer time.

The above destablizing effect of the lag L holds in general for the parameters located within regions in which the fundamental price is locally stable for lower lags and

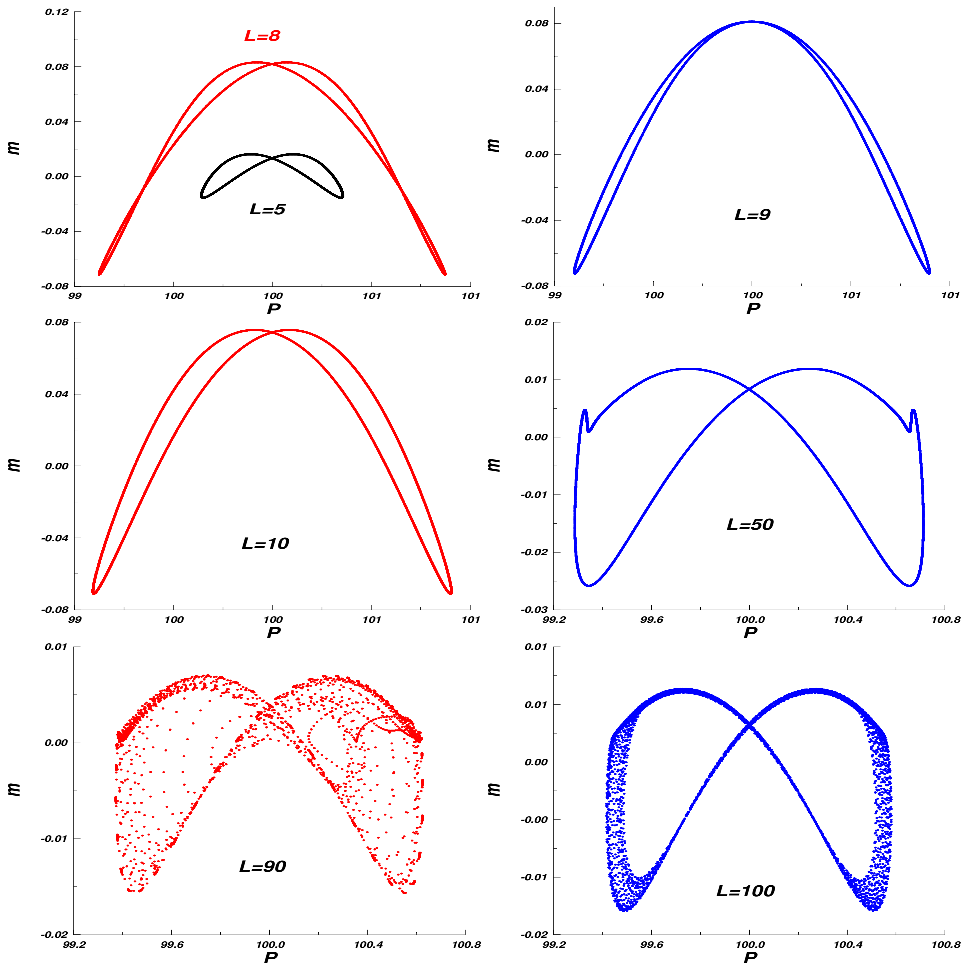

FIGURE 5.1. Phase plots of (m_t, P_t) for fixed for fixed $\beta = 0.4$ and various $L = 5, 8, 9, 10, 50, 90$ and 100.

unstable for higher lags, as discussed in the above. However, this may not always be the case. As a matter of fact, when the reaction coefficients from both types of traders are carefully balanced (such that $\bar{\alpha} = 1 + \bar{a}$), an increase of the lag length can stabilize an otherwise unstable system, as indicated in Corollary 3.3[7].

6. TIME SERIES ANALYSIS

The nonlinear dynamic model considered in the previous sections can be treated as the deterministic skeleton of the corresponding stochastic model. The price observed

[7]Numerical simulations (not reported here) indicate that, in this case, an increase in L can cause an explosive system to become a (locally) stable system.

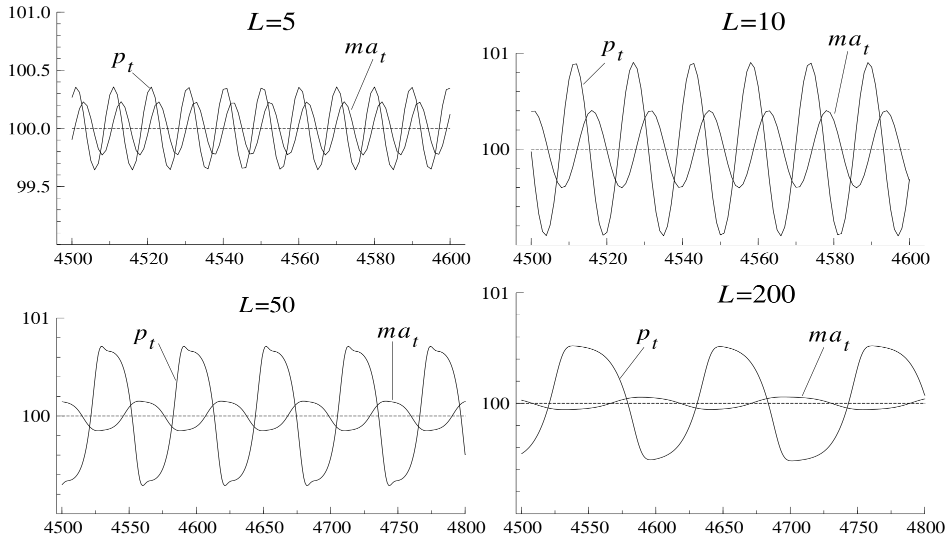

FIGURE 5.2. Price time series for fixed $\beta = 0.4$ and various $L = 5, 10, 50$ and 200.

in real markets are presumably the outcome of the interaction of both non-linear and stochastic elements. Rigourous analytical tools for the analysis of non-linear stochastic dynamical system are still in the development phase (see Arnold ((1998)) for the most up-to-date account). It seems difficult at the moment to carry out for the non-linear stochastic model the type of analysis that we have undertaken for the nonlinear deterministic model in previous section. In this section we shall mainly try to gain some insights into the behaviour of the nonlinear stochastic model through numerical simulations.

Apart from the noisy demand process introduced in Section 2, we also introduce a stationary random walk fundamental price process. The stationarity here means that the relative price change of the fundamental price follows a stationary $N(0, \sigma_\delta)$ process, namely,

$$P^*_{t+1} = P^*_t[1 + \sigma_\delta\, \delta_t], \tag{6.1}$$

where $\sigma_\delta \geq 0$ is a constant measuring the volatility of the relative return and $\delta_t \sim \mathcal{N}(0, 1)$. For illustration, we select

$$\alpha = 0.5, \beta = 0.3, a = 1, \eta = 0.2, C = 1, L = 100, P^* = P_0 = \$100.$$

To see the effect of the two noise processes on the price dynamics of the deterministic model, we compare four different cases in terms of $(\sigma_\epsilon, \sigma_\delta)$: (a) $(0, 0)$, (b)$(\sigma_\epsilon, 0)$, (c) $(0, \sigma_\delta)$ and (d) $(\sigma_\epsilon, \sigma_\delta)$ with $\sigma_\epsilon = 0.5\%$ and $\sigma_\delta = \sigma/K, \sigma = 5\%$ per annum and $K = 250$ (corresponding to 250 trading days per year). The comparison is conducted

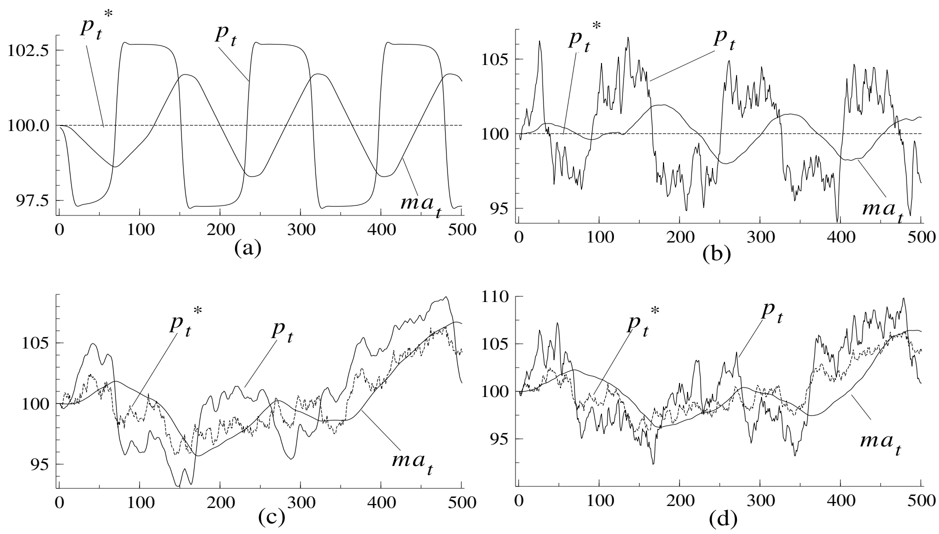

FIGURE 6.1. Time series of the market price p_t, the fundamental price P^* (for (a) and (b)) P_t^* (for (c) and (d)), and the moving average ma_t for fixed $L = 100$ with $(\sigma_\delta, \sigma_\epsilon) = (0, 0)$ for (a); $(0, 0.5\%)$ for (b); $(5\%p.a., 0)$ for (c) and $(5\%p.a., 0.5\%)$ for (d). Here $\alpha = 0.5, \beta = 0.3, \mu = 1, \eta = 0.2, a = 1, C = 1$.

over the first 500 time steps (a trading period of about 2 years). In all three cases, Fig. 6.1 compares the market price P_t, together with the fundamental price and the long-run moving average, Fig. 6.2 compares the difference of the market population fractions $m_t = n_{f,t} - n_{c,t}$, and Fig. 6.3 compares the demand functions of the fundamentalists and the technical analysts.

Case (a) reduces to the corresponding deterministic case. In this case, the constant fundamental price $P^* = 100$ is unstable (this may due to the strong extrapolation from the technical analysts) and the market price P_t displays a periodic switching between bull and bear markets, as illustrated in Fig. 6.1(a). From Fig.6.3(a), one can see that the fundamentalists and the technical analysts take opposite (long/short) positions in most of the time period. Because of limit position the technical analysts can take[8] and stability role of the fundamentalists, such off-setting positions cause the price to become bounded. However the market switches when both of them are in the same position and such a transition happens very quickly. In addition, the market is dominated by the technical analysts most of the time, as indicated by the fact that the trend of the market price in Fig. 6.1(a) follows closely the demand pattern of the

[8]This may be due to their short selling constraint when they hold a short position and consumption needs when they hold a long position.

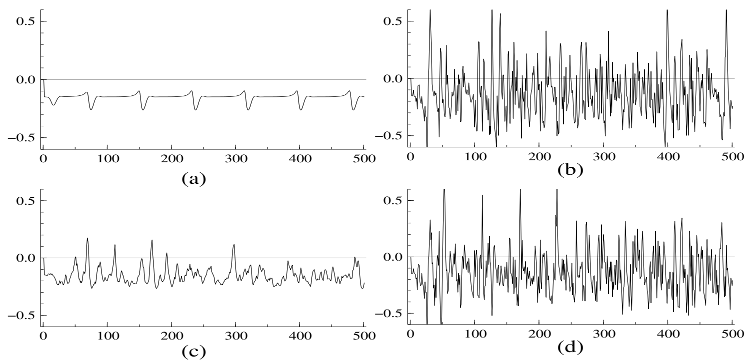

FIGURE 6.2. Time series of the market population fraction difference m_t for fixed $L = 100$ with $(\sigma_\delta, \sigma_\epsilon) = (0, 0)$ for (a); $(0, 0.5\%)$ for (b); $(5\%p.a., 0)$ for (c) and $(5\%p.a., 0.5\%)$ for (d). Here $\alpha = 0.5, \beta = 0.3, \mu = 1, \eta = 0.2, a = 1, C = 1$.

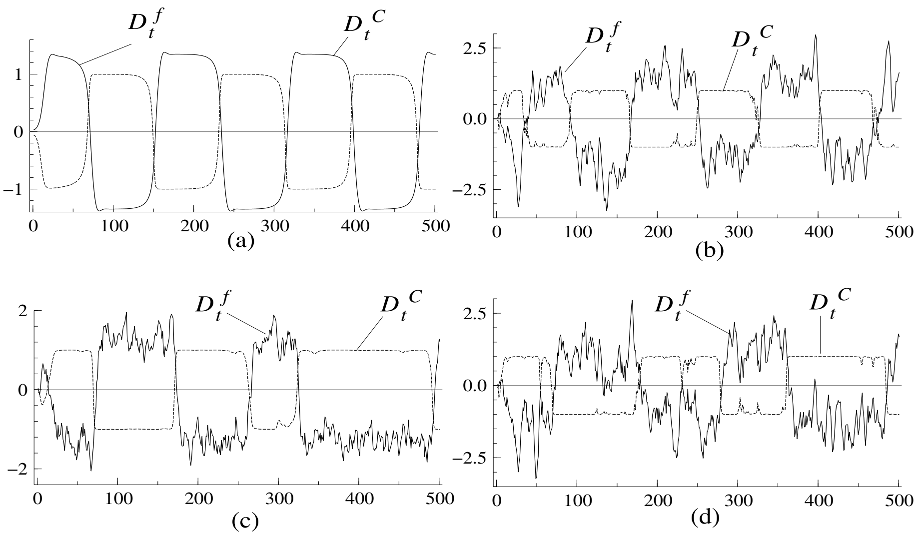

FIGURE 6.3. Time series of the demand functions of the fundamentalists D_t^f and the chartistsD_t^c for fixed $L = 100$ with $(\sigma_\delta, \sigma_\epsilon) = (0, 0)$ for (a); $(0, 0.5\%)$ for (b); $(5\%p.a., 0)$ for (c) and $(5\%p.a., 0.5\%)$ for (d). Here $\alpha = 0.5, \beta = 0.3, \mu = 1, \eta = 0.2, a = 1, C = 1$.

technical analysts in Fig. 6.3(a) and that traders tend to switch from the steady state level to the technical analysts as indicated by in Fig. 6.2(a).

Case (b) examines the effect of the noisy demand on the price dynamics. Because of this noisy demand, the market price becomes more volatile. However, the market price (in Fig.6.1(b)) and the demand functions (in Fig.6.3(b)) are dominated by the underlying price dynamics of the deterministic case (a), although the switching between two types of trading strategies is intensified (see Fig. 6.2(b)), spreading between $m = -60\%$ and $m = 60\%$.

Case (c) examines the effect of the noisy fundamental price on the price dynamics. One can see from Fig. 6.1(c) that the market price P_t closely follows the fundamental price P_t^*, though the variation of the market price increases (because of the strong extrapolation of the technical analysts). Fig. 6.2(c) shows that traders tend to switch to fundamentalist analysis from time to time. However, a comparison of the market price trend in Fig. 6.1(c) and the demand function pattern in Fig. 6.3(c) shows that the market price is above (below) the fundamental price when the technical analysts take long (short) position. This means the market price is still dominated by the technical analysts although it follows closely the fundamental price.

Case (d) examines the combined effect of the two noisy processes on the price dynamics. Apart from the fact that the market price becomes more volatile (because of the noisy demand), it shares similar features as in the cases (b) and (c). That is, the market price follows the fundamental price and the market is dominated by the technical analysts.

Based on the analysis above, we observe some interesting phenomena. (i) Adding the noisy demand can increase the price volatility, but it has less impact on the price pattern and the market conditions of the underlying price dynamics. (ii) When the fundamental price becomes more informative[9], the market price follows closely the fundamental price. (iii) The market is mainly dominated by the technical analysts (when they extrapolate strongly). They may be the winners over short time periods (indicated by the traders' switching to technical analysis), however over the whole time period they may be the losers in the sense that most of the time they buy when the market prices are high and sell when the market prices are low. (iv) The switching between bull and bear markets happens when both types of traders take the same position, a very intuitive result. Such transitions can be intensified with the help of the noise traders, leading to market bubbles and crashes.

7. CONCLUSIONS

Within the Brock and Hommes (1998) asset pricing model with heterogeneous and adaptive beliefs, price fluctuations are driven by an evolutionary dynamic system switching between different expectation schemes. Consequently various rational routes to randomness are observed when the intensity of choice to switch prediction strategies is high. This analytically oriented framework relies on the mathematical tractability of lower dimensional systems and it is in general difficult to have a clear

[9]This means that the fundamental price, which is known by the fundamentalists, follows a stochastic process, instead of a constant

picture when the prediction strategies involve a long history of price, such as the long-run moving average rules. Motivated by the popularity of moving averages strategies in both the real market and empirical studies, this paper sets out to analyze the impact of the moving average on the market dynamics and potentially rational routes to randomness. Within the confines of a model of the fundamentalists and technical analysts (who trade on the signals generated by the cross of the latest price over the long moving average) we are able to obtain some important qualitative insights into the impact of the moving average rule in general. Intuitively one might expect that a long moving average smooths price dynamics and hence an increase of the length of the moving average is expected to stabilize the market price dynamics. However our results show that, within a market maker scenario, this is in general not true (except when both the reaction coefficient α of the fundamentalists and the extrapolation rate a of the trend followers are balanced in certain way). In fact, the length of the moving average destabilises the market price and, to our knowledge, this is a new result related to the dynamics of the moving average rules. Another contribution of this paper is that similar rational routes to randomness occur when the switching intensity is high across various moving average rules. Time series analysis shows the potential of the model to explain various market phenomena such as price volatility, bull and bear markets and bubbles and crashes. In subsequent research, a more realistic model of the market with large number of trading rules, in particular with agents using different moving average strategies, should be studied extensively by using various numerical simulation tools, such as genetic algorithms and neural networks.

APPENDIX A. PROOFS OF MAIN RESULTS

A.1. **Proof of Proposition 3.1.** The deterministic system (2.15)-(2.17) can be written as follows:

$$\begin{cases} P_{t+1} &= F(X_t) \\ U_{t+1} &= H(X_t) \\ m_{t+1} &= G(X_t). \end{cases} \tag{A.1}$$

where

$$X_t = (P_t, P_{t-1}, \cdots, P_{t-(L-1)}, U_t, m_t),$$

$$F(X_t) = P_t + \frac{\beta}{2}[-(1-m_t)\alpha(P_t - P^*) + (1-m_t)h(\psi_t^{S,L})], \tag{A.2}$$

$$H(X_t) = [-\alpha(P_t - P^*] - h(\psi_t^{S,L})][F(X_t) - P_t] + \eta U_t, \tag{A.3}$$

$$G(X_t) = \tanh[\beta(H(X_t) - C)/2]. \tag{A.4}$$

One can easily see that, for $\eta \in [0, 1)$, $(P_t, U_t, m_t) = (P^*, 0, m^*)$ is the unique steady state of the system (A.1), where P^* corresponds to the constant fundamental price and $m^* = \tanh(-\beta C/2)$. Also, evaluated at the unique steady state,

$$\frac{\partial F}{\partial P_t} = 1 + \frac{\mu}{2}[-(1+m^*)\alpha + (1-m^*)a(\frac{1}{S} - \frac{1}{L})],$$

$$\frac{\partial F}{\partial P_{t-1}} = \frac{\partial F}{\partial P_{t-2}} = \cdots = \frac{\partial F}{\partial P_{t-(L-S)}} = \frac{\mu}{2}(1-m^*)a(\frac{1}{S} - \frac{1}{L}),$$

$$\frac{\partial F}{\partial P_{t-(L-S-1)}} = \cdots = \frac{\partial F}{\partial P_{t-(L-1)}} = \frac{\mu}{2}(1-m^*)a(-\frac{1}{L})$$

and

$$\frac{\partial F}{\partial U_t} = \frac{\partial F}{\partial m_t} = 0, \frac{\partial H}{\partial P_t} = \frac{\partial H}{\partial P_{t-1}} = \cdots = \frac{\partial H}{\partial P_{t-(L-1)}} = 0,$$
$$\frac{\partial H}{\partial U_t} = \eta, \frac{\partial H}{\partial m_t} = 0, \frac{\partial G}{\partial P_t} = \frac{\partial G}{\partial P_{t-1}} = \cdots = \frac{\partial G}{\partial P_{t-(L-1)}} = 0, \frac{\partial G}{\partial U_t} = \eta\beta/2, \frac{\partial G}{\partial m_t} = 0.$$

Based on these calculations, one can verify that the characteristic equation of the steady state has the form of (3.1).

A.2. **Proof of Lemma 3.2.** For $S = 1$ and $\bar{\alpha} = 1 + \bar{a}$,

$$\Gamma_{1L}(\lambda) \equiv \lambda^L + \frac{\bar{a}}{L}(\lambda^{L-1} + \cdots + \lambda + 1) = 0.$$

It follows from Chiarella and He (2002) that $|\lambda_i| < 1$ iff $-\frac{1}{L} < \frac{\bar{a}}{L} < 1$, i.e., $\bar{a} < L$ (since $\bar{a} > 0$).

In general, following from Jury's test, necessary conditions for $|\lambda_i| < 1$ for all i are $\bar{a}/L < 1$, $\Gamma_{iL}(1) = \bar{\alpha} > 0$ and

$$(-1)^L\Gamma_{1L}(-1) = \begin{cases} 2 - \bar{\alpha} + \bar{a} > 0 & \text{for } L = 2l \\ 2 - \bar{\alpha} + \frac{L+1}{L}\bar{a} > 0 & \text{for } L = 2l + 1 \end{cases}$$

A.3. **Proof of Proposition 3.4.** Let $f(\lambda) = \lambda^L$ and $g(\lambda) = -(1-\bar{\alpha}+\bar{a})\lambda^{L-1} + \frac{\bar{a}}{L}[\lambda^{L-1} + \cdots + \lambda + 1]$. Then on $|\lambda| = 1$,

$$|g(\lambda)| < |1 - \bar{\alpha} + \bar{a}| + \bar{a}, \quad |f(\lambda)| = 1.$$

If $1 + \bar{a} < \bar{\alpha} < 2$, then $|g(\lambda)| < |f(\lambda)|$ on $|\lambda| = 1$. Following from Rouche's theorem, $f(\lambda)$ and $\Gamma_{1L}(\lambda) = f(\lambda) + g(\lambda)$ have the same number of zeros inside $|\lambda| = 1$. Therefore $|\lambda_i| < 1$ for $i = 1, 2, \cdots, L$.

A.4. **Proof of Proposition 3.5.** For $S = 1$,

$$\Gamma_{1,L}(\lambda) \equiv \lambda^L - (1-\bar{\alpha})\lambda^{L-1} - \bar{a}(1 - \frac{1}{L})\lambda^{L-1} + \frac{\bar{a}}{L}(\lambda^{L-2} + \cdots + \lambda + 1) = 0.$$

i.e.

$$\Gamma_{1,L}(\lambda) \equiv \lambda^L - [1 - \bar{\alpha} + \bar{a}(1 - \frac{1}{L})]\lambda^{L-1} + \frac{\bar{a}}{L}(\lambda^{L-2} + \cdots + \lambda + 1) = 0.$$

- For $L = 1$,

  $$\Gamma_{1,L}(\lambda) \equiv \lambda - (1 - \bar{\alpha}) = 0.$$

  Hence $|\lambda| < 1$ iff $0 < \bar{\alpha} < 2$. Also $\lambda = +1$ for $\bar{\alpha} = 0$ and $\lambda = -1$ for $\bar{\alpha} = 2$.
- For $L = 2$,

  $$\Gamma_{2,1}(\lambda) = \lambda^2 + c_1\lambda + c_2 = 0,$$

  where

  $$c_1 = -(1 - \bar{\alpha} + \frac{1}{2}\bar{a}), \quad c_2 = \frac{\bar{a}}{2}.$$

  Following Jury's test, $|\lambda_i| < 1$ iff

  $$\begin{aligned} \pi_1 &\equiv 1 + c_1 + c_2 = \bar{\alpha} > 0, \\ \pi_2 &\equiv 1 - c_1 + c_2 = 2 - \bar{\alpha} + \bar{a} > 0, \\ \pi_3 &\equiv 1 - c_2 = 1 - \frac{\bar{a}}{2} > 0. \end{aligned}$$

  Hence P^* is LAS if $(\bar{\alpha}, \bar{a}) \in D_{12}(\bar{\alpha}, \bar{a})$. Also, $\lambda_1 = 1$ and $|\lambda_2| < 1$ when $\pi_1 = 0$, $\lambda_1 = -1$, $|\lambda_2| < 1$ when $\pi_2 = 0$ and $\lambda_{1,2} \in C$, $|\lambda_{1,2}| = 1$ when $\pi_3 = 0$.
- For $L = 3$,

  $$\Gamma_{1,3}(\lambda) \equiv \lambda^3 - [1 - \bar{\alpha} + \bar{a}(1 - \frac{1}{3})]\lambda^2 + \frac{\bar{a}}{3}(\lambda + 1) = 0.$$

  Denote $c_1 = -[1 - \bar{\alpha} + \frac{2}{3}\bar{a}]$, $c_2 = c_3 = \frac{\bar{a}}{3}$. Then $|\lambda_i| < 1$ iff

  $$\begin{aligned} \pi_1 &\equiv 1 + c_1 + c_2 + c_3 = \bar{\alpha} > 0, \\ \pi_2 &\equiv 1 - c_1 + c_2 - c_3 = 2 - \bar{\alpha} + \frac{2}{3}\bar{a} > 0, \\ \pi_3 &\equiv 1 - c_2 + c_1c_3 - c_3^2 = 1 - \frac{\bar{a}}{3}[2 - \bar{\alpha} + \bar{a}] > 0. \end{aligned}$$

  Hence P^* is LAS if $(\bar{\alpha}, \bar{a}) \in D_{13}(\bar{\alpha}, \bar{a})$. Furthermore, $\pi_1 = 0$, $\pi_2 = 0$ and $\pi_3 = 0$ give the saddle-node, flip and Hopf bifurcation boundaries, respectively.

- For $L = 4$,

$$\Gamma_{1,4}(\lambda) \equiv \lambda^4 - [1 - \bar{\alpha} + \frac{3}{4}\bar{a}]\lambda^3 + \frac{\bar{a}}{4}(\lambda^2 + \lambda + 1) = 0.$$

Denote $p = -[1 - \bar{\alpha} + \frac{3}{4}\bar{a}], q = \frac{\bar{a}}{4}$. Then, using Jury's test, $|\lambda_i| < 1$ iff $\Gamma_{1,4}(1) = \bar{\alpha} > 0, \Gamma_{1,4}(-1) = 2 - \bar{\alpha} + \bar{a} > 0, \bar{a} < 4$ and both the determinants of matrixes

$$A = \begin{pmatrix} 1 & 0 & q \\ p-1 & 1+q & 0 \\ 2q-p & p-1 & 1+p-q \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 0 & -q \\ p & 1-q & -q \\ 0 & p-q & 1-p \end{pmatrix}$$

are positive. It can be verified that $|A| > 0, |B| > 0$ iff $(1+q)^2[1+p-2q] + q(p-1)^2 > 0$ and $p < 1$, respectively, which leads to the result.

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